Enhanced MAD for Real Option Valuation

and the Application of Market Utility

A Case Study of Abandonment Option

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Abstract

In sharp contrast to the dominant real option valuation that assumes a stochastic process for an investment's capital value, this paper demonstrates the valuation of a real option assuming that cash flow follows a stochastic process. We show that this method is at least equally effective and sometimes more intuitive. We note that, in a discounted cash flow (DCF) framework, only when certain constraints are met, assuming capital value as a geometric Brownian motion (GBM) is compatible with simultaneously assuming cash flow as a GBM; otherwise, making both assumptions in the valuation would lead to an incorrect real option value. We clarify the above argument with a simple textbook-standard case study.

We further extend the traditional valuation approach using a market utility, which we illustrate in the same case study. We show that it provides a simpler valuation process and in particular, overcomes the challenge of changing the measure in traditional approaches. We further argue that this approach provides more flexibility and a clearer rationale for individual investors in their decision making.

Key words: real option, decision making, investment opportunity, geometric Brownian motion, abandonment option, marketed asset disclaimer, change of measure, lease, rental value, market utility, risk tolerance, risk aversion

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1 Introduction

We implement the marketed asset disclaimer (MAD) approach proposed by Copeland and Antikarov (2001) in a numerical case reported by Azevedo-Pereira (2001) in Howell et al. (2001), hereafter referred to the Campeiro case. In particular, we value a real option to abandon a real estate lease that is not associated with a market price. In contrast to the original MAD approach which values the abandonment option based on assumptions about the gross lease value evolution, we take cash flows (rents) as the process that is driving the uncertainty in the gross lease value. We then explain how to derive a value for the abandonment option that is consistent with the value obtained under the original MAD approach.

Irrespective of whether it is the investment value or cash flow that is taken as the driving source of uncertainty, we have to change from the physical ($\mathbb{P}$) measure for the investment valuation to the risk-neutral ($\mathbb{Q}$) measure for the real option valuation. This means that we need to derive the risk-neutral probability under which the gross investment value follows a $\mathbb{Q}$-martingale. The consistent valuation of the underlying investment in the $\mathbb{P}$ measure with the real option valuation in the $\mathbb{Q}$ measure presents some complexities which have been ignored by previous papers that consider the MAD approach. Indeed, Azevedo-Pereira applies a $\mathbb{Q}$ measure for the abandonment option valuation that is not consistent with the $\mathbb{P}$ measure for the lease value.[1]

Under the standard MAD approach, it is the evolution of the gross investment value specified in $\mathbb{P}$ and translated into $\mathbb{Q}$, that determines the real option value under $\mathbb{Q}$. However, in the Campeiro case, the gross lease value process in $\mathbb{P}$ is derived from the assumption of a constant risk-adjusted discount rate for the *net* cash flow, which implies that the risk-adjusted discount rate for the *gross* cash flow is time-varying. Such calculations can be quite complex.

Motivated by the complexity that must be introduced to the MAD approach in order to

[1] List other papers that apply the MAD approach with unconsistent $\mathbb{Q}$ and $\mathbb{P}$ measure include …

derive the correct risk-neutral measure for the real option valuation, this paper proposes an extension of the MAD approach which values the real option under the same $\mathbb{P}$ measure as that applied to value the base investment. For this, we follow Kasanen and Trigeorgis (1994) by deriving the utility function for the "market representative investor" and applying this utility to derive the real option value as the certainty equivalent. The advantage of this approach is that a single utility can be implied from all investments in the market, even if they have different risk-adjusted discount rates for deriving their net present values. This utility can be applied to value all real options or more investments in the same market. By contrast, when the MAD approach is applied to multiple assets, each risk-adjusted discount rate could generate a different equivalent $\mathbb{Q}$ measure.

In the following, Section 2 briefly describes the MAD approach and then introduces the settings of our extension. We make assumptions about the evolution of cash flow or investment value, and in both cases we use the MAD approach to value the abandonment option and derive the assumptions about the dynamics of cash flows and the dynamics of investment values for which the abandonment option value is nearly identical. Section **??** derives general conditions that must be satisfied by the expected gross and net investment value at any time before the real option expires, so as to link the DCF analysis on base investment and the valuation of the attached real option under RNV. Following the conditions, we exemplify the derivations of the risk-adjusted discount rate for cash flow if missing and the dividend yield if assuming the cash flows are the dividend paid out from the investment. More importantly, we show the compatibility of assuming that either the cash flows or investment value evolves according to a GBM, under those general conditions. In order to numerically illustrate our findings, we introduce the Campeiro case in Section 3 and value the real option consistently. Section 4 numerically and intuitively explain the problem about different and mostly simpler approximations of (a) the risk-adjusted discount rate for cash flows if missing; (b) the drift of the gross investment value process if assuming to be a GBM; and (c) the risk-neutral probabilities derived from the DCF analysis on the base investment for the later valuation of the attached real option. Section 5 proposes the market utility-based extension of the MAD approach and demonstrates its applications to the Campeiro case. Section 6 concludes.

2 An Extended MAD Approach

The original MAD approach comprises the separate valuation of the base investment and the attached real option; the former employs a conventional DCF analysis, and the latter adopts the standard risk-neutral financial option valuation technique. To justify this approach, Copeland and Antikarov argue that DCF analysis provides the best estimate of the current value for the base investment when the market price is unobservable, and that the attached real option can be perfectly hedged using the base investment, so its valuation should be carried out under the risk-neutral measure.[2]

Copeland and Antikarov's approach is limited to the case where data on the risk-adjusted discount rate for cash flows is known. The DCF analysis estimates the gross (net) investment value at time 0, and the risk-neutral valuation of the option requires the evolution of the gross investment value in the risk-neutral measure $\mathbb{Q}$. Given that the point value of this process under the physical measure $\mathbb{P}$ is the discounted sum of all future cash flows at that point, we would at least be able to specify the gross investment value process under $\mathbb{P}$ if we knew the risk-adjusted discount rate for cash flows. However, in practice, very often there are no data which allow this discount rate to be obtained, and instead, the risk-adjusted discount rate for net income is used as a proxy.[3]

[2]The argument seems widely accepted. For instance, in the asset pricing model by Jagannathan and Wang (1996). Another example can be found in enterprise valuation for mergers and acquisitions (M&A); an enterprise is valued as the portfolio of the asset-in-place and the attached real options (see Lambrecht, 2004). not so many words but lots of references.

[3]For instance, in order to value at enterprise, one can download the company-specific weighted average cost of capital (WACC) from bloomberg, and use it to discount the earnings before interest after tax (EBIAT); where EBIAT is the net income and WACC is the risk-adjusted discount rate for net income.

We propose an extension of the MAD approach which is more generally applicable. To motivate our ideas we begin with a brief summary of the original MAD approach which also serves to introduce some notation.

2.1 The Original Approach of Copeland and Antikarov

Copeland and Antikarov first calculate q_0, the gross value of the base investment at time 0 excluding any attached real option. It is the sum of the present values of all future expected cash flows from the investment. For the present value, each expected cash flow is discounted by its required risk-adjusted discount rate.[4] Hence, in continuous time,

$$q_0 = \int_0^T \mathbb{E}^{\mathbb{P}}[x_\tau] \exp\left(-\int_0^\tau \tilde{r}_s ds\right) d\tau, \tag{1}$$

where T denotes the investment maturity, x_τ is the investment cash flow for $0 \leq \tau \leq T$, and $\tilde{r}_s$ represents the deterministic risk-adjusted discount rate for cash flow at time s, $0 \leq s \leq \tau$.[5] Regarding the investment cost, c_0, Copeland and Antikarov assume that a lump-sum payment in advance is required. Correspondingly, p_0, the net investment value at time 0 excluding any attached real option, is the present gross investment value less the cost, *viz.*

$$p_0 = q_0 - c_0. \tag{2}$$

For the real option valuation, Copeland and Antikarov assume that q_t, the gross investment value at time t evolves over time according to a GBM under the $\mathbb{P}$ measure, *i.e.*

$$\frac{dq_t}{q_t} = (\nu_t - \delta_t)\, dt + \eta_t dB_t, \tag{3}$$

[4] Note that, the risk-adjusted discount rate in (1) captures the uncertainty associated with only the future cash flows but not the investor's abandonment of these cash flows.

[5] Remark on notation: we write $\mathbb{E}^{\mathbb{P}}[x_\tau] := \mathbb{E}_0^{\mathbb{P}}[x_\tau]$, where $\mathbb{E}_i^{\mathbb{P}}[x_\tau]$ denotes the expected value of x_τ at time i, $0 \leq i \leq \tau$ under the $\mathbb{P}$ measure that is defined for the future value of x_τ.

with ν_t and η_t being deterministic, q_0 given by (1), and the cash flow is generated from the investment as dividends:

$$x_t = q_t\tilde{\delta}_t, \quad \tilde{\delta}_t = 1 - \exp\left(-\delta_t\right) \ , \tag{4}$$

where δ_t ($\tilde{\delta}_t$) is the dividend yield.[6] To pin down the process of the gross investment value, they compute the dividend yield as follows

$$\tilde{\delta}_t = \frac{\mathbb{E}^{\mathbb{P}}[x_t]}{\mathbb{E}^{\mathbb{P}}[q_t]}, \tag{5}$$

where, using the DCF analysis, they define

$$\mathbb{E}^{\mathbb{P}}[q_t] = \int_t^T \mathbb{E}^{\mathbb{P}}[x_\tau] \exp\left(-\int_t^\tau \tilde{r}_s ds\right) d\tau, \tag{6}$$

and

$$\mathbb{E}^{\mathbb{P}}[x_t] = x_0 \exp\left(\mu_t t\right), \tag{7}$$

where μ_t is deterministic. Copeland and Antikarov then change from the $\mathbb{P}$ measure to the $\mathbb{Q}$ measure, under which the gross investment value process (3) is effectively changed to

$$\frac{dp_t}{p_t} = (r_t - \delta_t)dt + \eta_t dB_t^0, \quad dB_t^0 = dB_t + \frac{\nu_t - r_t}{\eta_t}dt. \tag{8}$$

Under this $\mathbb{Q}$ measure, they value the real option using the standard option pricing technique.

2.2 The Extended Approach

The first generalisation of our approach is to replace the lump-sum cost assumption (c_0) by allowing deterministic, periodic cost payments for the investment, simultaneous with his receipt of cash flows. We denote by k_t the investment cost at time t, and by y_t the net income, *i.e.*

[6] $\tilde{\delta}_t$ is equivalent to δ_t but discretely compounded.

$y_t = x_t - k_t$. This means the investment effectively generates net income, periodically, before its expiry. Now, the net investment value at time 0 can alternatively be computed as the sum of the present values of all future expected net incomes, and the total investment cost at time 0 is the sum of all discounted future periodic costs. That is,

$$p_0 = \int_0^T \mathbb{E}^{\mathbb{P}}[y_\tau] \exp\left(- \int_0^\tau \hat{r}_s ds \right) d\tau, \tag{9}$$

where $\hat{r}_s$ denotes the deterministic risk-adjusted discount rate for net income at time s; and

$$c_0 = \int_0^T k_\tau \exp\left(- \int_0^\tau r_s ds \right) d\tau, \tag{10}$$

with r_s being the deterministic risk-free rate at time s. By no-arbitrage the gross (net) investment value at time 0 is unique. This means, p_0, calculated using (9), should be identical to (2), where c_0 in (2) is either a lump-sum payment or set by (10). Equivalently, q_0 (1) should be equal to q_0 in (2).

Now we model the gross investment value at time t under the $\mathbb{P}$ measure. In particular, we can assume a gross investment value process which guarantees the expected cash flows (7). Alternatively, we can assume a cash flow process with the expectation (7) and then the gross investment value is a discounted sum of all expected cash flows in the future. Nevertheless, let us first describe the relationship between the gross investment value and the cash flow as follows,

$$q_t = x_t Y(\cdot) \ , \tag{11}$$

where $Y(\cdot)$ is a deterministic function, as the gross investment value is perfectly correlated with the cash flow - see (1).[7]

[7] The net investment value and the net income are interdependent as well, according to (9).

Given that the expected investment values must satisfy the following condition,

$$\mathbb{E}^{\mathbb{P}}[q_t] = \mathbb{E}^{\mathbb{P}}[p_t] + c_t, \tag{12}$$

where, from the DCF analysis,[8]

$$\mathbb{E}^{\mathbb{P}}[p_t] = \int_t^T \mathbb{E}^{\mathbb{P}}[y_\tau] \exp\left(-\int_t^\tau \hat{r}_s ds\right) d\tau, \quad c_t = \int_t^T k_\tau \exp\left(-\int_t^\tau r_s ds\right) d\tau,$$

we can derive the $Y(\cdot)$ function as either

$$Y(t) = \int_t^T \exp\left(\int_t^\tau (\mu_s - \hat{r}_s) ds\right) d\tau + \int_t^T y(\tau, t) \left[\exp\left(-\int_t^\tau r_s ds\right) - \exp\left(-\int_t^\tau \hat{r}_s ds\right)\right] d\tau \ ,$$

with $y(\tau, t) = k_\tau^{-1} x_0 \exp(\mu_t t)$, or

$$Y(t) = \int_t^T \exp\left(\int_t^\tau (\mu_s - \tilde{r}_s) ds\right) d\tau.$$

Hence, if we assume a process for the gross investment value and treat the cash flows as dividends, the dividend yield, $\tilde{\delta}_t$, by definition (5), must be equal to the reciprocal of $Y(t)$ and that,[9]

$$\delta_t = -\log\left(1 - Y(t)^{-1}\right) \ . \tag{13}$$

For instance, we can follow Copeland and Antikarov and assume a GBM (3) evolution of the

[8] Intuitively, for a real option that is similar to an American vanilla call, the gross investment value q_t and the investment cost c_t are equivalent to the underlying share price and the (time-varying, deterministic) strike of the financial option, and the pay-off to the option, if exercised, is the net investment value p_t, *viz.* $p_t = q_t - c_t$.

[9] Consider the case where the investment cost shares the same source of uncertainty as the net income ($r \to \hat{r}$). The dividend yield would be independent with the investment cost. Especially, when the drift of the cash flow is also equal to the risk-adjusted discount rate for cash flow ($\mu = \hat{r}$), the time 0 expectation of the gross investment value at time t defined by (6) becomes $\mathbb{E}^{\mathbb{P}}[q_t] = (T - t)\mathbb{E}^{\mathbb{P}}[x_t]$. Now, the dividend yield by definition, would only depend on time. We can also show this by computing the limit of $\tilde{\delta}_t$ (13) as $r, \mu \to \hat{r}$: $\lim_{r,\mu \to \hat{r}} \tilde{\delta}_t = (T - t)^{-1}$.

gross investment value. The drift of the process can then be specified using (13):

$$\nu_t - \delta_t = \mu_t + \frac{Y(t)'}{Y(t)} \ , \tag{14}$$

where $Y(t)' = dY(t)/dt$.[10] Alternatively, we can assume a cash flow process, *e.g.* a GBM:

$$\frac{dx_t}{x_t} = \mu_t dt + \sigma_t dB_t \ . \tag{15}$$

Then the evolution of the gross investment value can be derived from (11); which form of $Y(t)$ to use depends on which discount rate (either $\tilde{r}_t$ or $\hat{r}_t$) we can obtain from the market data. Note that, (15) is actually compatible to (3) - under (15), the gross investment value process is also a GBM with the same drift as in (14):[11]

$$\frac{dq_t}{q_t} = \left(\mu_t + \frac{Y(t)'}{Y(t)}\right) dt + \sigma_t dB_t, \tag{16}$$

and that the dispersions of the cash flow process and the gross investment value process are identical, *i.e.* for the same investment,

$$\sigma_t = \eta_t. \tag{17}$$

In addition, when $\tilde{r}_t$ is unobtainable but $\hat{r}_t$ is trackable, if we must use the first form of $Y(t)$, we can compute $\tilde{r}_t$ using $\hat{r}_t$:[12]

$$\tilde{r}_t = \hat{r}_t - G(M_t, t), \tag{18}$$

[10] $dx_t = d\left(q_t Y(t)^{-1}\right) = q_t Y(t)^{-1}\frac{dq_t}{q_t} - q_t Y(t)^{-1}\frac{dY(t)}{Y(t)} = \left(\nu_t - \delta_t - \frac{Y(t)'}{Y(t)}\right) x_t dt + \eta_t x_t dB_t$. Now, we apply (7) and hence, $\nu_t - \delta_t - \frac{Y(t)'}{Y(t)} = \mu_t$.

[11] $dq_t = d\left(x_t Y(t)\right) = x_t Y(t)\frac{dx_t}{x_t} + x_t Y(t)\frac{dY(t)}{Y(t)} = \left(\mu_t + \frac{Y(t)'}{Y(t)}\right) q_t dt + \eta_t q_t dB_t$.

[12] This equation is obtaned by solving (12) recursively.

where M_t denote the moneyness at time t of the real option attached to the investment, and the adjustment $G(\cdot)$ takes the form:

$$G(M_t, t) = \log\left(1 + \frac{g(t)}{\mathbb{E}^{\mathbb{P}}[M_t]}\right), \quad \mathbb{E}^{\mathbb{P}}[M_t] = \frac{\mathbb{E}^{\mathbb{P}}[q_t]}{c_t} = \frac{\mathbb{E}^{\mathbb{P}}[p_t]}{c_t} + 1,$$

with $g(t) = \exp(\hat{r}_t - r_t) - 1$.

(18) shows that using $\hat{r}_t$ as a proxy of $\tilde{r}_t$ is not appropriate, because the former is different from the latter unless the investment cost is close to zero ($\mathbb{E}^{\mathbb{P}}[M_t] \to \infty$), or it becomes stochastic and shares the same source of uncertainty with the net income ($r_t \to \hat{r}_t$).[13] Indeed, net income, by definition, is perfectly correlated with the cash flow. Therefore, intuitively the risk-adjusted discount rate for cash flow is not equal to that for net income.[14] Furthermore, $\tilde{r}_t$ may significantly diverge from $\hat{r}_t$ and yield to r_t for the investment whose total cost is close to its gross value ($\mathbb{E}^{\mathbb{P}}[M_t] \to 1$).[15] Intuitively, when the expected future cash flows from the investment are only enough to cover the total cost, the investment would be associated with very little risk, otherwise it would immediately be abandoned by the holder or unwanted in the market.

Now given the process of the gross investment value (11) in $\mathbb{P}$ measure, we look for the an equivalent $\mathbb{Q}$ measure (under which the discounted gross investment value is a martingale as in 8). That is, we keep the gross investment value at any time t the same under both measure (according to the law of one price) and derive the risk-neutral probabilities, with which the expected gross investment value at any time t only grows at the risk-free rate over time, *viz.*

$$\mathbb{E}_i^{\mathbb{Q}}[q_t] = q_i \exp\left(\int_i^t (r_\tau - \delta_\tau) d\tau\right), \quad 0 \le i \le t. \tag{19}$$

[13] $\lim\limits_{r_t \to \hat{r}_t} g = 0 \Rightarrow \lim\limits_{r_t \to \hat{r}_t} G(M_t) = 0 \Rightarrow \lim\limits_{r_t \to \hat{r}_t} \tilde{r}_t = \hat{r}_t + \lim\limits_{r_t \to \hat{r}_t} G(M_t) = \hat{r}_t$.

[14] Consider the case when k_t is constant, then the change in cash flow between time t_1 and t_2 is identical to that in net income in the same time period; however, the relative changes are different, and this is the relative change which determine the risk-adjusted discount rate.

[15] $\lim\limits_{\mathbb{E}^{\mathbb{P}}[M_t] \to 1} G(M_t) = r_t - \hat{r}_t \Rightarrow \lim\limits_{\mathbb{E}^{\mathbb{P}}[M_t] \to 1} \tilde{r}_t = r_t$.

We can then value the real option under RNV.

In the following, we numerically demonstrate how to find the process of the gross investment value as well as how to change the measure for the real option valuation. For this, we introduce the Campeiro case by Azevedo-Pereira (2001).

3 The Campeiro Case

The Campeiro case concerns an investor who holds a real estate lease for T years, during which time the investor receives regular rental payments from a tenant and has regular maintenance costs associated with the upkeep of the property. The annual maintenance cost is assumed to be constant, denoted by k, and the uncertainty which drives the lease value and the abandonment option value lies only in the amount of the rental income received during the year. After T years the lease expires and the investor has no further claim on the property.

At the beginning of each year $t \leq T$. the investor has a real option, since he can abandon the lease at no cost, thus saving the annual maintenance cost but also forgoing the annual rent during years $(t+1), \ldots, T$. Clearly the investor would abandon the lease if all his future rents were less than the future maintenance costs, otherwise he would make a loss on this lease.

We refer to the rent received during year t as x_t, which is expected to grow at a fixed annual rate $\mu = 16.848\%$ with a variation $\sigma = 32.208\%$. At time 0, the rent $x_0 = 6.8182$. Furthermore, the risk-free rate $r = 9.531\%$, and the risk-adjusted discount rate associated with the net income $\hat{r} = 25.254\%$.[16] In addition, we call the rent x_t less the maintenance cost $k = 3$ the net income y_t. In the Campeiro case, the lease is held for 10 years. Note that rent and cost are assumed to occur annually in advance so that $T = 9$ and we index years using $t = 0, 1, \ldots, T$.

[16] Although, following Azevedo-Pereira, we employ a discrete-time framework, we remark that we use continuous-time compounding. The annual interest rates defined in the Campeiro case are appropriately re-defined here so that the effective annual rates are identical, for example a 10% annual rate is expressed as 9.531% continuously compounded.

3.1 The Process of Gross Lease Value

We can now calculate the gross lease value at time 0. However, we cannot apply (1) for this, because the risk-adjusted discount rate for rent $\tilde{r}_t$ is unknown. Instead, we hold that p_0 and c_0 in (2) is given by (9) and (10) respectively. That is, in discrete time,

$$p_0 = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{P}}[y_\tau] \exp(-\hat{r}\tau) = \sum_{\tau=0}^{T} (x_0 \exp(\mu\tau) - k) \exp(-\hat{r}\tau) = 35.7130 \ ,$$

and

$$c_0 = \sum_{\tau=0}^{T} \left\{ k e^{-r\tau} \right\} = 20.2771.$$

Hence, q_0 in (2) can be computed as the sum of these two values,

$$q_0 = p_0 + c_0 = 35.7130 + 20.2771 = 55.9901. \tag{20}$$

Now, let us derive the process of the gross lease value. Following the vast majority of papers on the MAD approach, we employ the Cox, Ross, and Rubinstein (1979) binomial tree parameterisation to discretise Azevedo-Pereira's assumption about the rent process (15). Regarding the binomial tree for rent, the time between successive nodes is $\Delta t = 1$ since rents and costs occur annually. In each time period, the underlying can move up by a factor $u > 1$ or down by another factor $d < 1$, and we denote the state at time t by $s(t) = 0$, u, d, ud, uu, du, $\ldots$, $uduu$ etc. In particular,

$$u = e^{\sigma\sqrt{\Delta t}} = 1.38, \quad d = u^{-1} = 0.725, \quad \pi_x = \frac{\exp(\mu\Delta t) - d}{u - d} = 70\%, \tag{21}$$

where π_x denotes the physical transition probability of $x_{s(t)}$ moving up at time t.[17] Recall the

[17] For instance, if the rent moves up at every step from time 0 until the option expiry year, then $x_{s(T)} = x_0 u^9 = 123.7600$, where $s(T) = \underbrace{uuu\ldots uu}_{9}$. At time 0, the probability of rent having this value at time T is $\pi_x^9 = 4.035\%$

compatibility of (3) and (15) that we clarified in Section 2. Hence, if we alternatively assume that the gross lease value follows a GBM (3) and build the binomial tree for it, the values of u and d that we should use are the same as above, since $\eta_t = \sigma = 32.208\%$ (see 17).[18]

Under the assumption (15), in order to derive the process of the gross lease value, we first need the binomial tree for rent, that is constructed as below,

$$x_{s(t)u} = x_{s(t)}u, \quad x_{s(t)d} = x_{s(t)}d, \tag{22}$$

and also the risk-adjusted discount rate for rent $\tilde{r}_t$. Given the constant risk-adjusted discount rate for net income $\hat{r}$ and the risk-free rate r, we calculate $\tilde{r}_t$ using (18).[19] With its values (see Part 1 of Table 1), we can now build the binomial tree for gross lease value using (11), where the time stamps of the gross investment value and the rent in (11) are replaced by their states.[20]

$$q_{s(t)} = x_{s(t)} \sum_{\tau=t}^{T} \exp\left(\mu(T-\tau) - \sum_{s=t}^{\tau} (\tilde{r}_s - \tilde{r}_t)\right). \tag{23}$$

Under the alternative assumption (3), we can construct the binomial tree for gross lease value as follows.

$$q_{s(t)u} = q_{s(t)}e^{-\delta_t}u, \quad q_{s(t)d} = q_{s(t)}e^{-\delta_t}d, \tag{24}$$

which starts from q_0 (20). For this, we require the dividend yield δ_t. In particular, we apply

under the $\mathbb{P}$ measure.

[18]Note that, the physical transition probability of $q_{s(t)}$ moving up at time t

$$\pi_{q,t} = \frac{\exp(\nu_t \Delta t) - d}{u-d} \neq \pi_x,$$

as v_t is time dependent (see 14). For now, we do not need to calculate ν_t and consequently $\pi_{q,t}$.

[19]This formula can also be derived from the following backward induction in discrete time:

$$q_{s(t)} = x_{s(t)} + \frac{\mathbb{E}^{\mathbb{P}}_{s(t)}\left[q_{s(t+1)}\right]}{\exp(\tilde{r}_{t+1})}, \quad \mathbb{E}^{\mathbb{P}}_{s(t)}\left[q_{s(t+1)}\right] = \pi_x q_{s(t)u} + (1-\pi_x)\, q_{s(t)d}\ ,$$

which starts at $t = 9$ with $q_{s(10)} := 0$ and ends at $t = 0$.

[20]To build a binomial tree for net lease values, denoted by $p_{s(t)}$, we deduct the total future costs at time t from each value in the gross lease value tree at the same time. That is, $p_{s(t)} = q_{s(t)} - k\int_t^T \exp(-r(\tau - t))d\tau$.

(13) with $Y(t)$ discretised for the Campeiro case:

$$Y(t) = \int_t^T \exp((\mu - \hat{r})(\tau - t))d\tau + \frac{k}{x_0 \exp(\mu t)} \int_t^T \exp(-r(\tau - t)) - \exp(-\hat{r}(\tau - t))d\tau.$$

The values of $\tilde{\delta}_t$ that we obtain are presented in Part 2 of Table 1.[21] Following this calculation, the binomial tree for gross lease value (24) can therefore be quantified.

Table 1: The values of the risk-adjusted discount rate for rents and the dividend yield for deriving the process of the gross lease value.

(a) The risk-adjusted discount rate required for rents that is derived from its relationship with the risk-adjusted discount rate for net income (18).

t	0	1	2	3	4	5	6	7	8	9
$\tilde{r}_t$	0	20.007%	20.672%	21.269%	21.799%	22.267%	22.679%	23.039%	23.352%	23.623%

(b) The dividend yields that is derived as a ratio between the expected rent and expected gross lease value according to (13).

t	0	1	2	3	4	5	6	7	8	9
$\tilde{\delta}_t$	12.18%	13.43%	14.94%	16.80%	19.22%	22.54%	27.44%	35.55%	51.69%	100.00%

3.2 The Risk-neutral Probabilities

Given the process of the gross lease value that we pin down under either (3) or (15) in the previous subsection, we now compute the risk-neutral probabilities for the abandonment option valuation.[22] For this, we first present (19) in discrete time as follows,

$$q_{s(t)} - x_{s(t)} = \mathbb{E}^{\mathbb{Q}}_{s(t)} \left[q_{s(t+1)} \right] \exp(-r), \quad \mathbb{E}^{\mathbb{Q}}_{s(t)} \left[q_{s(t+1)} \right] = \pi^{\mathbb{Q}}_t q_{s(t)u} + \left(1 - \pi^{\mathbb{Q}}_t \right) q_{s(t)d} \ ,$$

[21] Corresponding to the binomial tree for gross lease value, we can easily construct a binomial tree for rents using the definition of rent (4).

[22] Note that, Azevedo-Pereira made the assumption (15) but computed the risk-neutral probabilities with which the discounted rent (rather than the discounted gross lease value) was a martingale. See Section 4.1 for further discussion of this important point.

where $s(t+1)$ denotes the succeeding states of $s(t)$: $s(t)u$ and $s(t)d$. We then solve this equation for $\pi_t^{\mathbb{Q}}$, and thus obtain the expression below.

$$\pi_t^{\mathbb{Q}} = \frac{\left(q_{s(t)} - x_{s(t)}\right)\exp(r) - q_{s(t)d}}{q_{s(t)u} - q_{s(t)d}}. \tag{25}$$

In accordance, under (15), we compute the values of $\pi_t^{\mathbb{Q}}$ for all time periods and present them in Table 2;[23] when assumption (3) is alternatively made, $\pi_t^{\mathbb{Q}}$ becomes a fixed number: $\pi_t^{\mathbb{Q}} = 57.252\%$.[24]

Table 2: The risk-neutral probability (25) such that the discounted gross lease value follows a $\mathbb{Q}$-martingale, assuming (15).

t	0	1	2	3	4	5	6	7	8
$\pi_t^{\mathbb{Q}}$	0.5203	0.5095	0.4999	0.4914	0.4839	0.4774	0.4717	0.4668	0.4625

So far we have calculated the risk-neutral probabilities under either assumption. The binomial tree for rent under (15) is quantified. Under (3), this tree can easily be constructed by definition (4). Now, let us consider the abandonment option valuation under RNV.[25]

3.3 Abandonment Option Valuation under RNV

Irrespective of whether we assume (3) or (15), the valuation process of the abandonment option is as follows. We can value the abandonment option either alone or together with the lease. Intuitively, if the future rent is less than the future cost, we would abandon the lease, in which case we save the costs but lose the further rents; otherwise we do not exercise the option, then it would become valueless. Hence, we can present the pay-off of the option with the lease as

[23] Under assumption (15), given the binomial trees for gross lease value (23) and for rent (22), we can simplify the above formula and present $\pi_t^{\mathbb{Q}}$ as a function of the risk-adjusted discount rate for $\tilde{r}_{t+1}$ and other constant parameters.

$$\pi_t^{\mathbb{Q}} = \frac{\exp(r + \tilde{r}_{t+1} - \mu) - d}{u - d}, \tag{26}$$

where $\pi_t^{\mathbb{Q}}$ would be time dependent, since the risk-adjusted discount rate for rent $\tilde{r}_{t+1}$ varies over time.

[24] Note that, under either assumption, the rent process is not a $\mathbb{Q}$-martingale.

[25] Instead of using the binomial tree for rent in the following abandonment option valuation, we can alternatively employ the binomial tree for net lease value which can be constructed according to Footnote 20 under either of assumptions (3) and (15).

$\max\{p_t, 0\}$ and without the lease as $\max\{0, -p_t\}$. This means that the option with and without the lease are analogous to an American call and a put respectively.

To calculate the net lease value with the option at time 0 under the $\mathbb{Q}$ measure, we apply the following backward induction.

$$C_{s(t)} = \max\left\{y_{s(t)} + \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[C_{s(t+1)}\right]\exp(-r), 0\right\}, \quad \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[C_{s(t+1)}\right] = \pi^{\mathbb{Q}}_t C_{s(t)u} + \left(1 - \pi^{\mathbb{Q}}_t\right) C_{s(t)d} \,, \tag{27}$$

where $C_{s(t)}$ is the net value of the lease and the option in state $s(t)$ under the $\mathbb{Q}$ measure. This backward induction starts from the last step at $t = 9$ with $C_{s(10)} := 0$ and goes backwards in time.[26] The abandonment option will have the value $P_0 = C_0 - p_0$.

Alternatively, we value the abandonment option alone using backward induction:

$$P_{s(t)} = \max\left\{-p_{s(t)}, \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[P_{s(t+1)}\right]\exp(-r)\right\}, \quad \mathbb{E}^{\mathbb{Q}}_{s(t)}\left[P_{s(t+1)}\right] = \pi^{\mathbb{Q}}_t P_{s(t)u} + \left(1 - \pi^{\mathbb{Q}}_t\right) P_{s(t)d} \,, \tag{28}$$

where $P_{s(t)}$ denotes the value of the option only at state $s(t)$ under the $\mathbb{Q}$ measure. This backward induction starts from $t = 9$ with $P_{s(10)} := 0$. To apply this backward induction, we would require the binomial tree for net lease value, which can be obtained using Footnote 20, regardless of assuming (3) or (15).

According to the put-call parity (PCP), the above two valuation processes are equivalent and hence, under the same assumption, they would generate identical abandonment option values (see Appendix A for further discussion). We present our calculation results in Table 3 along with the corresponding assumptions. There is a minor difference between the option values under (3) and (15). We believe that it is only a discretisation error. With more steps in the binomial

[26] For instance, $C_{s(9)} = \max\left\{y_{s(9)}, 0\right\}$. It is then simple to use; $C_{s(8)} = \max\left\{y_{s(8)} + \mathbb{E}^{\mathbb{Q}}_{s(8)}\left[C_{s(9)}\right]\exp(-r), 0\right\}$ where $\mathbb{E}^{\mathbb{Q}}_{s(8)}\left[C_{s(9)}\right] = \pi^{\mathbb{Q}}_8 C_{s(8)u} + \left(1 - \pi^{\mathbb{Q}}_8\right) C_{s(8)d}$. The rest of this backward induction follows.

Table 3: A summary of the assumptions (3) and (15) and the corresponding abandonment option values. The relative option values are in percentage and equal to the abandonment option values proportional to the net lease value (20).

	Main Assumption	Option price	Relative option price
(15)	x_t follows a GBM	0.3998	1.120%
(3)	q_t follows a GBM	0.4040	1.131%

trees that we construct, the two option values should be the same.

So far we have demonstrated the valuation of a real option with an assumed stochastic process for the evolution of either the cash flow or the investment value. The impression that practitioners are sometimes given is that only the latter assumption should be used to value the real option. In contrast, we hold that either assumption can be appropriate to use in specific instances. In an efficient rental market (or any market) where the tenants (investors) are predominantly interested in rents (investment cash flows), we might recommend to value real options assuming (15). In the conditions where the cash flows are significant and observable, this assumption may also be preferred. On the other hand, one would prefer to value the real option by employing (3) if, referring to a lease (or any other investments), one has not only an abandonment option but other real options related to the lease (investments) price such as an option to sell the lease, to share the ownership of the lease, or to redevelop the lease for other use.

4 Alternative Approximations of MAD Parameters

We have approximated the parameters consistently and valued the abandonment option under (3) and (15). We have also shown that the variation of assumptions would lead to the changes in both the valuation of the option and its value. Regarding the ways of approximating $\pi_t^{\mathbb{Q}}$, $\tilde{r}_t$, and $\tilde{\mu}_t$, the change effect on the option value could also be significant. For instance, in Azevedo-Pereira's study of the Campeiro case, $\pi_t^{\mathbb{Q}}$ was mis-calculated and hence, it led to a significantly different and obviously incorrect value of the abandonment option.

In the following, we first explain Azevedo-Pereira's valuation of the abandonment option. This allows us to illustrate the importance of changing the measure consistently in the valuation. We later choose different approximations of $\tilde{r}_t$ and $\tilde{\mu}_t$. It will then be clear that our approximations regarding these parameters are the most advanced ones.

4.1 A Discussion of Azevedo-Pereira's Abandonment Option Valuation

In the original study of Campeiro case, Azevedo-Pereira obtained the value 12.3505 for the abandonment option. That this is different from our option value is due to the inconsistent change-of-measure in his calculation. In this subsection, we exhibit Azevedo-Pereira's assumptions and calculations, so as to illustrate the impact on the real option valuation regarding the change-of-measure.

Under the $\mathbb{P}$ measure, Azevedo-Pereira firstly assumed (15) and then employed the Cox, Ross, and Rubinstein binomial tree parameterisation to construct the tree for rents. The values that he chose for u, d and π_x were identical with ours (21), so was his calculation of the net lease value at time 0 (20). However, for the option valuation, in contrast with $\pi_t^{\mathbb{Q}}$ (26), he applied a different risk-neutral probability.[27]

$$\pi^{\mathbb{Q}'} = \frac{\exp(r) - d}{u - d} = 57.252\% \ ,$$

where $\mathbb{Q}'$ denotes a risk-neutral measure which differs from $\mathbb{Q}$.

Now the problem is that the $\mathbb{Q}'$ measure is inconsistent with the $\mathbb{P}$ measure. To see this, we show that the gross lease value at time 0 diverges from q_0 (20), which is against the law of one

[27] With this transition probability, clearly the discounted rent would be a $\mathbb{Q}'$-martingale over time.

price.[28]

$$q_0' = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{Q}'}[x_\tau] \exp(-r\tau) = (T+1)x_0 = 68.1818 \ .$$

We therefore argue that it is inappropriate to value the abandonment option under the $\mathbb{Q}'$ measure. We illustrate this point by analysing Azevedo-Pereira's option valuation. He applied (27) with $\pi^{\mathbb{Q}'}$ to the tree of rents and obtained the net lease value with option as $C_0^{\mathbb{Q}'} = 48.0635$. He then claimed the option value to be $C_0^{\mathbb{Q}'} - p_0 = 48.0635 - 35.7130 = 12.3505$.

Yet this option value is too high to be correct, as the abandonment option is a deep out-of-the-money put. The option strike is the total costs of all time periods $c_0 = 20.2771$ (20), whereas the current underlying price is the gross lease value at time 0, either 55.9901 or 68.1818; the latter value is much higher. Intuitively, it can hardly seem plausible that this option is worth roughly 20% of the current underlying price.

In fact, the number 12.3505 comprises not only the abandonment option value but also the change in the current lease value due to Azevedo-Pereira's inconsistent change-of-measure from $\mathbb{P}$ to $\mathbb{Q}'$. This value change can be calculated as the difference between the current gross (net) lease values under the two measures $q_0' - q_0 = p_0' - p_0 = 12.1917$.[29]

To conclude, when we value a real option under RNV whereas the underlying investment valuation is under some physical measure, an inconsistent change-of-measure would bias the real option value.

4.2 Assuming $\tilde{r}_t$ under (15) Is Constant

Under (15), in order to value the abandonment option, recall that we apply (18) to calculate the time-varying risk-adjusted discount rate for rent $\tilde{r}_t$ such that we obtain the process of the gross

[28] We can also calculate the net lease value at time 0: $p_0' = \sum_{\tau=0}^{T} \mathbb{E}^{\mathbb{Q}'}[y_\tau] \exp(-r\tau) = 47.9047$, which is also different from (20).

[29] Note that, since the maintenance costs are risk-free under any measure, they would always have the same values regardless of the measures they are under. Therefore $q_0' - q_0 = (p_0' - c_0) - (p_0 - c_0) = p_0' - p_0$, and we can use either the gross or net lease value to detemine the change in the current lease value due to the change-of-measure.

lease value (23). A simpler way is to derive a constant risk-adjusted discount rate, denoted by $\tilde{r}$, that sets (2) and (9) equal. This is to say,

$$(\exp(-\tilde{r}))^{T+1-t} - \tilde{\delta}_t^{-1}(\exp(-\tilde{r})) + \tilde{\delta}_t^{-1} - 1 = 0, t = 0 \ .$$

However, we note that, this approximation is flawed. Because for any time $t > 0$, this equation would not hold for a fixed value of $\tilde{r}$ and consequently, (12) no longer holds. Hence, the abandonment option would be mis-priced.

Let us calculate the abandonment option value so that we can observe how much difference this approximation can make. Bringing in the inputs of the Campeiro case, we obtain $\tilde{r} = 21.416\%$. The risk-neutral probability (25) is then $\pi^{\mathbb{Q}} = 49.752\%$ and hence, the value of the abandonment option is 0.3774. This value is almost 10% lower than that calculated with the time-varying $\tilde{r}_t$ (18), 0.3998.

We explain this mis-valuation intuitively as follows. In contrast with $\tilde{r}_t$, we see that the constant risk-adjusted discount rate for rent $\tilde{r}$ overvalues the rents from $t = 4$ to maturity, which already have higher expectations than the short term rents ($t = 1$, 2 and 3), whilst the current rent x_0 remains the same. That is to say, if the investor abandon the lease, he would lose the long term rents and only obtain the short term ones. Hence, he would tend to wait for future rents to cover the maintenance costs than abandoning the lease early. The abandonment option therefore becomes less attractive and consequently undervalued.

4.3 Assuming $\tilde{\mu}_t$ under (3) Is Fixed

To simplify the calculation under (3), we may assume a fixed drift ν instead of a time-varying ν_t (14). However, we note that this approximation would fail (12) and therefore would lead to a slightly flawed abandonment option value.

To see this, let us calculate the abandonment option value first. Given (7), we solve for the dividend yield $\tilde{\delta}_t$ and ν the system of non-linear equations presented as below.

$$\tilde{\delta}_t = \frac{\tilde{\delta}_{t-1}}{1 - \tilde{\delta}_{t-1}} \exp(\mu - \nu).$$

with $\tilde{\delta}_0 = x_0/q_0$ and $\tilde{\delta}_9 = 1$. Bringing in all other inputs, we present the values of $\tilde{\delta}_t$ as in Table 4. The associated risk-neutral probability (26) under (3) applies here. The abandonment option

Table 4: The values of dividend yield when the process of gross lease value has a fixed drift ν.

τ	0	1	2	3	4	5	6	7	8	9
$\hat{\delta_\tau}$	12.18%	13.25%	14.59%	16.32%	18.63%	21.87%	26.74%	34.87%	51.14%	100.00%

value is thus $0.3796 < 0.4040$. This under-estimation occurs because the gross lease value $q_{s(t)}$ is in general overvalued. To see this, we compare the dividend yields reported in Table 4 to 1(b). For the same expected rent at time t, the dividend yield at the same time in Table 4 is lower than that in 1(b). This means the gross lease value that we calculate here, as the base of the dividend yield, is higher than that calculated in previous valuation; hence, the investor would believe in a higher profitability of the lease and therefore be less attempted to abandon the lease. Equivalently speaking, the abandonment option is undervalued.

5 An Alternative of Abandonment Option Valuation under RNV

In this section, we extend the MAD approach to derive a real option value, consistent with Copeland and Antikarov's RNV, but based entirely within the same $\mathbb{P}$ measure as that defined by the process of investment value (or cash flow). To do this in the traditional DCF framework, we need to solve the problem that there are two different risk-adjusted discount rates are required, one for the base investment value (or its cash flows) and another for valuing the real option, however, there is no information that allows one to derive the second risk-adjusted discount rate - the real option pay-off cannot be observed in the market as a risk premium. The lack of a unique representation for risk-adjusted discount rates that apply to all real options on a given investment motivates the introduction of a market utility-based extension of the MAD approach.[30]

Furthermore, there are some complications in valuing a real option under RNV. In particular, the implementation of MAD under (15) has shown that it can be difficult to find appropriate risk-neutral probabilities for the real option valuation. The methodological error in Azevedo-Pereira (2001)'s valuation is also associated with the change-of-measure issue. Under (3) it is easier to find the $\mathbb{Q}$ measure although the theoretical reasoning underlying the derivation is less secure. It is always difficult to estimate the expectation and volatility of the underlying investment value. These problems remain unresolved in the extant real option literature where there is no observable underlying investment market price. In order to bypass these complications, we can extend MAD by adopting a market utility basis, with which we obviate the need for the $\mathbb{Q}$ measure and use the same $\mathbb{P}$ measure as in the investment valuation.

The effectiveness of market utility has already been demonstrated by Kasanen and Trigeorgis (1994). In particular, we adopt a model design proposed by Smith and Nau (1995) and assume that the utility applied in the valuation is exponential and additively separable. We then apply the utility-based MAD approach to value the abandonment option in Campeiro case. Despite its restrictive assumption regarding the utility function and possible estimation errors of parameters,

[30] In practice, individual investors model the risk-adjusted discount rate for a single investment relative to other investments in the same market. For instance, one of the compositions of the risk-adjusted discount rate is the required premium for the investment's systematic risk. Commonly, it is calculated relative to the required risk premium for a market portfolio comprised of investments that are popular in the market.

However, the risk-adjusted discount rate models that are used by various investors may include parameters that can only be estimated subjectively, due to the lack of observations in the market. For instance, when a company values the target firm for a potential acquisition, it may have to calculate the risk-adjusted discount rate associated with the target firm's synergy mainly based on the experience of its previous acquisitions as well as its knowledge regarding that target firm. Therefore, even with the same risk-free rate for all investors and for the same investment, the equivalent $\mathbb{Q}$ measure that an investor derives from his calculation of the risk-adjusted discount rate may differ from those that others derive.

Similarly, for a single investor and with an observable risk-free rate in the market, since the risk-adjusted discount rates for different investments are modelled involving some subjective estimation, the $\mathbb{Q}$ measures that are derived from these risk-adjusted discount rates may not be unique and therefore cannot be applied to value the real options attached to all the investments.

In contrast, we propose to calibrate the utility function of the "market representative investor" using the values of multiple investments that are already agreed by the market. We can then apply this function to value the attached real options and more investments in the market.

we are able to show that this extended MAD approach is in line with the traditional MAD approach.

5.1 The Case for a Market Utility-Based Extension of MAD

The received literature applies RNV, with very restrictive assumptions, to price a real option in order to use a simple valuation process. However, as we have demonstrated, the valuation can be more complicated than it at first appears. Alternatively, we can value a real option using the decision-tree analysis (DTA) approach. The problem of this latter method is, as pointed out by Copeland and Antikarov, the difficulty of estimating the risk premium of any contingent claim attached to real investment(s). Here we propose a more general solution in which we apply a market-based utility in the DTA. The argument is as follows.

The risk premium of any investment or contingent claim should correspond to its risk which is defined in the appropriate asset pricing theory. One obvious candidate model might, for instance, be the capital asset pricing model (CAPM). We can always value a contingent claim or real investment in a market from the perspective of the representative investor of that market. Therefore, instead of estimating the risk premium of a specific real option or investment, we may determine the risk preference of the market representative investor with which the prices of all contingent claims and real investments in that market can be determined and therefore arbitrage-free. The benefit of this method is that the risk preference of the market representative investor can always be applied when the premia of a particular real investment or attached contingent claim is unknown.

We are not arguing against the view that in many cases the risk-neutral valuation can be very convenient. However, if the valuation of a real option has to be consistent with the valuation of the underlying investment based on its market risk premium, we propose to value the real option, just as valuing the underlying investment, according to the market representative investor's risk preference but in a utility-based framework, rather than under an equivalent risk-neutral measure.

Furthermore, this extension can later be integrated with other utility-based decision analyses; real options that are characterised more by the idiosyncratic risk than market risk can still be valued.

5.2 Abandonment Option Valuation Based on Market Utility

The market utility-based valuation model of real option valuation was implemented by Kasanen and Trigeorgis who showed the effectiveness and consistency of the method when compared with RNV. If we can back out the market risk preferences based on the prices of the most liquid investments in a market, as shown by Kasanen and Trigeorgis, we can then use this, regardless of the assumptions of the cash flows (investment value) process we make, to value any contingent claim or other investment in that market.

For the market utility function, for simplicity, we hereby consider two basic and popular one-parameter utility functions: exponential and power utilities, both of which were used by Kasanen and Trigeorgis. We denote by γ the relative risk aversion of the market representative investor and the only parameter required in the market utility function. We also arbitrarily assume that the initial wealth of the representative investor $\omega_0 = 100$.

In order to value the lease and the abandonment option based on the market utility, we adopt the assumption that the utility is additively separable. It is also used in much of the real option valuation literature (see for instance Smith and Nau, 1995). With this assumption, the net lease value is calculated using:

$$(p_{s(t)})_0 = \left(x_{s(t)} - k\right)_0 + \mathrm{CE}\left(\mathbb{E}^{\mathbb{P}}\left[U\left(p_{s(t+1)}\right)_0\right]\right),$$

$$\mathbb{E}^{\mathbb{P}}\left[U\left(p_{s(t+1)}\right)_0\right] = \pi_t^{\mathbb{P}} U\left(p_{s(t)u}\right)_0 + \left(1 - \pi_t^{\mathbb{P}}\right) U\left(p_{s(t)d}\right)_0 \ .$$

where $U(x)$ is the utility of a certain amount of wealth x and $\mathrm{CE}\left(U(x)\right)$ is the certainty equivalent of the utility value $U(x)$. Note that all the values we put in the utility functions are in present value terms. The notation $(x)_0$ represents the value of x in time 0 terms (discounted at the risk-free rate r). This backward induction starts from $t = 9$ with $\left(p_{s(10)}\right)_0 := 0$. The resultant net lease value $(p_0)_0$ should be equal to p_0 (20) according to the law of one price. By equating these two net lease values we can then back out the market risk aversion γ.

We can now value the abandonment option using the utility function and risk aversion applied in (derived from) the previous valuation of the lease. Furthermore, in comparison with Section 3.3 which is under RNV, we value the abandonment option by altering (27) and (28) to

$$C^{\mathbb{P}}_{s(t)} = \max\left\{\left(\left(x_{s(t)} - k\right)_0 + \mathrm{CE}\left(\mathbb{E}^{\mathbb{P}}\left[U\left(C^{\mathbb{P}}_{s(t+1)}\right)\right]\right)\right), 0\right\},$$

$$\mathbb{E}^{\mathbb{P}}\left[U\left(C^{\mathbb{P}}_{s(t+1)}\right)\right] = \pi^{\mathbb{P}}_t U\left(C^{\mathbb{P}}_{s(t)u}\right) + \left(1 - \pi^{\mathbb{P}}_t\right) U\left(C^{\mathbb{P}}_{s(t)d}\right) \ .$$

and

$$P^{\mathbb{P}}_{s(t)} = \max\left\{\left(-p_{s(t)}\right)_0, \mathrm{CE}\left(\mathbb{E}^{\mathbb{P}}\left[U\left(P^{\mathbb{P}}_{s(t+1)}\right)\right]\right)\right\},$$

$$\mathbb{E}^{\mathbb{P}}\left[U\left(P^{\mathbb{P}}_{s(t+1)}\right)\right] = \pi^{\mathbb{P}}_t U\left(P^{\mathbb{P}}_{s(t)u}\right) + \left(1 - \pi^{\mathbb{P}}_t\right) U\left(P^{\mathbb{P}}_{s(t)d}\right) \ .$$

The abandonment option price is then $P_{s(0),\mathbb{P}}$ or equivalently the premium in $C_{s(0),\mathbb{P}}$ above p_0 (20). The equivalence between these two calculations of the abandonment option price is shown can be derived similarly as in Appendix A with the assumption that the utilities applied in the calculation is additively separable.

The resultant option prices based on various utilities with calibrated γ are listed below. In general, we see that the market utility-based prices of the abandonment option are similar to the risk-neutral prices. The minor differences would appear to be caused by the restrictive

Table 5: The abandonment option price calculated using our market utility-based extended MAD under (3) or (15), mainly that the rents (gross lease value) follow a GBM. The market risk aversion γ is calibrated assuming $p_0 = 35.7130$. Initial wealth $\omega_0 = 100$.

	Main assumption	MAD	Market risk aversion	Option price
(15)	x_τ follows a GBM	RNV	-	0.4094
		$-\exp(-\gamma x/\omega_0)$	8.8770	0.3057
		$-(x/\omega_0+1)^{1-\gamma}$	12.0000	0.4413
(3)	q_τ follows a GBM	RNV	-	0.4040
		$-\exp(-\gamma x/\omega_0)$	5.4112	0.3656
		$-(x/\omega_0+1)^{1-\gamma}$	6.0000	0.3822

assumption of additively separable utility, or more importantly, the estimation error of γ. Note that in this case study, we rely only on the net value of one investment - the lease - to back out the market risk aversion. The results could be improved if we could calibrate the market risk aversion by taking into account of the prices of more and particularly more liquid investments in the market.

6 Conclusion

In contrast to the lack of enthusiasm in practice to value real options using MAD based on an assumed evolution of cash flows, this paper shows such a valuation process of a real option is equivalent and consistent with the traditional implementation of MAD, *i.e.* valuing a real option according to an assumed stochastic process of the investment values. More precisely, when cash flows and investment value share one source of uncertainty, the evolution of the cash flows (investment values) would determine the process of the investment values (cash flows). For instance, when the cash flows follow a simple GBM, we show how this leads to a straightforward derivation of the implied investment value. Based on either of these two processes, we also demonstrate the valuation of the real option attached to the investment.

By valuing the real option with various assumptions, this paper demonstrates the link between the value of the real option and the valuation of the underlying investment *via* the change-of-measure. The most classic and popular operation in practice regarding the valuation of an investment is conducted in the DCF framework, whilst the paradigm of the real option valuation literature and practice is within RNV framework. Switching from the DCF framework to the risk-neutral world requires a change-of-measure, mainly the derivation of the appropriate risk-neutral probabilities.

Yet the change-of-measure may be problematic to achieve in practice. For instance, before an acquisition, an investment bank may fund the acquirer having calculated the value of the target company, produced in the DCF framework (albeit heavily adjusted or extended). Then the acquirer may simply take this single value and price its own real option to acquire without running through and being consistent with the assumptions made in the valuation of the target company conducted by the investment bank.

On the other hand, the danger of applying inappropriate risk-neutral probabilities in the real option valuation, is significant and can result in a mis-estimated real option value. A typical example is the real option valuation in the original case study by Azevedo-Pereira (2001), which is examined in Section 4.1. The mis-estimation is shown to be too large to be ignored - Azevedo-Pereira (2001) valued the abandonment option as worth 12.3505 which is roughly a third of the net lease value. This is an implausible result for a deep out-of-the-money American put option and is shown here to be more than twenty times larger than the correct value(s).

This paper also demonstrates the market utility-based valuation of the real option. It is at least as efficient when comparied with the real option valuation under RNV principle. The the market utility-based real option value could, however, change with the risk aversion of the representative investor. The sensitivity of the real option value with respect to the risk aversion has been the subject of considerable discussion in the literature. See for instance Henderson (2002, 2007).

But once we adopt a utility-based valuation model, it opens up various intriguing possibilities for exploration. We can, for example, see how much a real option is worth to an individual investor whose subjective risk preferences differ systematically from the market. This means, apart from calculating its market price, an investor may also check the sensitivity of the real option value with respect to the market risk preferences and compare his risk preferences to the market risk attitude so as to determine how valuable the real option is to him given his subjective view regarding the underlying investment.

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A The Equivalence of the Option Valuation Processes (27) and (28)

The option valuation processes (27) and (28) are equivalent according to put-call parity (PCP):

$$P_{s(t)} + p_{s(t)} = C_{s(t)} \ . \tag{29}$$

The proof is as follows.

We start with backward induction (28) at time 0. That is,

$$P_0 = \max\left\{(-p_0)\,, \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r)\right\} \ .$$

Adding p_0 to both sides of the equation,

$$P_0 + p_0 = \max\left\{(-p_0 + p_0)\,, \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r) + p_0\right\} \ . \tag{30}$$

We now rewrite the second term in the maximisation,

$$\mathbb{E}^{\mathbb{Q}}\left[P_{s(1)}\right]\exp(-r) + p_0 = \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)} + p_{s(1)}\right]\exp(-r) - \mathbb{E}^{\mathbb{Q}}\left[p_{s(1)}\right]\exp(-r) + p_0 \ ,$$

where, bringing back the relationship between the gross and net lease value given in Footnote 20,

$$-\mathbb{E}^{\mathbb{Q}}\left[p_{s(1)}\right]\exp(-r) + p_0 = -\mathbb{E}^{\mathbb{Q}}\left[q_{s(1)} - k\sum_{\tau=1}^{T} e^{-r(\tau-1)}\right]\exp(-r) + \left(q_0 - k\sum_{\tau=0}^{T}\exp(-r\tau)\right) \ .$$

After some simple algebra, we rewrite this expression as,

$$-\mathbb{E}^{\mathbb{Q}}\left[q_{s(1)} - k\sum_{\tau=1}^{T}\exp(-r\tau)\right]\exp(-r) + \left(q_0 - k\sum_{\tau=0}^{T}\exp(-r\tau)\right) = \left(q_0 - \mathbb{E}^{\mathbb{Q}}\left[q_{s(1)}\right]\exp(-r)\right) - k \ ,$$

where, assuming it is q_t that follows a $\mathbb{Q}$ martingale (25),

$$q_0 - \mathbb{E}^{\mathbb{Q}}\left[q_{s(1)}\right]\exp(-r) = x_0 \ .$$

With these decompositions, we can rearrange (30) as following,

$$P_0 + p_0 = \max\left\{0, \mathbb{E}^{\mathbb{Q}}\left[P_{s(1)} + p_{s(1)}\right] + (x_0 - k)\right\} \ . \tag{31}$$

Now apply the put-call parity (29) to (31), we reach to

$$C_0 = \max\left\{0, \left(\mathbb{E}^{\mathbb{Q}}\left[C_{s(1)}\right] + x_0 - k\right)\right\},$$

which is indeed backward induction (27) at time 0.

This proof applies at any $t \in [0, T]$.